

# **Promoting the understanding of representations by grade 3 pupils: Two teachers' practices**

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*In this paper we analyse the practice of two grade 3 teachers in order to understand how they promote their pupils' understanding of representations. Two lessons were video recorded and we focused on teacher-pupils' interactions. Data collected from three different moments (introduction of the task, pupils' autonomous work, and whole class discussion) were analysed through content analysis. The results show that the way the teachers organize the different classroom moments are related to their perception of pupils' needs and difficulties and that to promote their pupils' understanding of representations both adapt their actions and questioning to this perception.*

*Keywords: teachers' practices, teachers' actions, representations.*

## **INTRODUCTION**

The way teachers use representations in their practice has a great influence in pupils' understanding of representations (Stylianou, 2010). Faced with a challenging situation it may be very difficult to pupils to choose a suitable representation to handle that situation. In addition, the fact that mathematical representations are related to each other enhances pupils' difficulties in understanding and learning about representations (Goldin, 2008). Tripathi (2008) suggests that teachers must use several types of representations to promote pupils' understanding of a given concept. However, Acevedo Nistal et al. (2009) refer that the use of too many representations may be in the origin of pupils' difficulties in making a suitable choice. In this study we look at the practice of two grade 3 teachers aiming to understand how they promote their pupils' understanding of representations.

## **TEACHERS' PRACTICES AND REPRESENTATIONS**

"Representation" includes the process of representing as well as the resulting product (NCTM, 2000). Bruner (1999) categorizes representations as active, iconic or symbolic. Thomas, Mulligan and Goldin (2002) refer three types of representations: pictorial, iconic, and notational. Webb et al. (2008) categorize representations in informal, preformal and formal and Ponte and Serrazina (2000) also refer to oral language.

The importance of the role of the teacher in supporting pupils' learning of representations has received attention from several authors. For example, McClain (2000) analyses a grade 1 teacher's practice, showing how she translates what pupils say into symbolic representations and how it influences the development of pupils' notations and symbolizations. According to the author, the teacher tries to get her pupils to use more formal representations, introducing the notation of addition and subtraction based on their answers. She concludes that the representations proposed by the teacher were gradually adopted and adapted by her pupils, contributing to the enrichment of whole group discussions. In a similar perspective, Stylianou (2010) refers to teachers' introduction of representations as a way to feature new concepts, illustrations and processes in solving problems. She states that creating links between these concepts is a crucial element to support pupils' learning. She suggests that teachers should use more than one representation related to the same concept, selecting those that they find more adequate.

For Swan (2007), the success of a task varies according to teachers' actions, how teachers lead pupils in doing it, the role that they assume, how they introduce the task, and the questions that they make during the whole class discussion. Teacher's actions can be analyzed regarding how they promote pupils' understanding of representations while they are involved in different kinds of activity, namely choosing or designing a representation, using and transforming a representation, or reflecting about representations (Table 1).

<b>Students' activity regarding representations</b>	<b>Teachers' actions</b>
Designing/Choosing	Promoting free choice
	Hinting through questioning
	Providing explicit suggestions or examples
Using/Transforming	Challenge students through open questioning
	Asking to explain in a structured way
	Guiding to establish connections
	Guiding to make conversions or treatments
Reflecting	Promoting the evaluation of the work done
	Promoting systematizations
	Informing

**Table 1 – Teachers' actions in different moments of the pupils' activity.**

Thus, to support their pupils' design or selection of a representation, teachers may (i) promote free choice, by letting them to decide about which the most appropriate representations are; (ii) hint pupils about the representation they should use; or (iii) give suggestions or examples. To promote pupils' use or transformation of a given representation, the teacher may (i) pose them open questions to make them think about hypothetical transformations (conversions or treatments) of representations; (ii) ask pupils in a more structured way to explain what they did; (iii) guide pupils to

establish connections between representations; or (iv) guide pupils to make conversions and treatments of representations. Finally, teachers may enhance their pupils' reflection about representations by (i) promoting the evaluation of the work that has been done; (ii) systematizing information; or (iii) informing about new representations and their connections with used ones.

## **METHODOLOGY**

This study is part of a qualitative research on the practices of primary school teachers concerning their work with mathematical representations and was undertaken in a school near Lisbon with the first author as a non-participant observer. The participants are Sofia and Sara, two young grade 3 teachers. They were on a team of grade 3 teachers that worked together very often. In this paper we present some episodes of their classes, showing how they strive to promote pupils' understanding of representation as they work on the following task: "In a theatre play performed by grade 3 pupils, João, Pedro and Ulisses wanted to be the King. On the other hand, Ana, Inês and Estrela wanted to play the Queen. How many pairs of King/Queen may be formed?"

The teachers chose this task taking into account that they felt that their pupils were struggling with problem solving with whole numbers. Data was collected by video recording and by collecting pupils' written work. We analysed data through content analysis in the moments of introduction of the task, pupils' autonomous work and whole class discussion (Ponte, 2005). Pupils' representations were categorized according to Bruner (1999), Thomas et al. (2002), Webb et al. (2008) and Ponte and Serrazina (2000). We categorized as informal representations all pictorial representations (drawings), as preformal we consider iconic representations (non mathematical symbols and schemes) and verbal representations (words) and as formal representation the symbolic representations (mathematical symbols). The teachers' actions were categorized according to the framework indicated in Figure 1.

## **SOFIA'S CLASS**

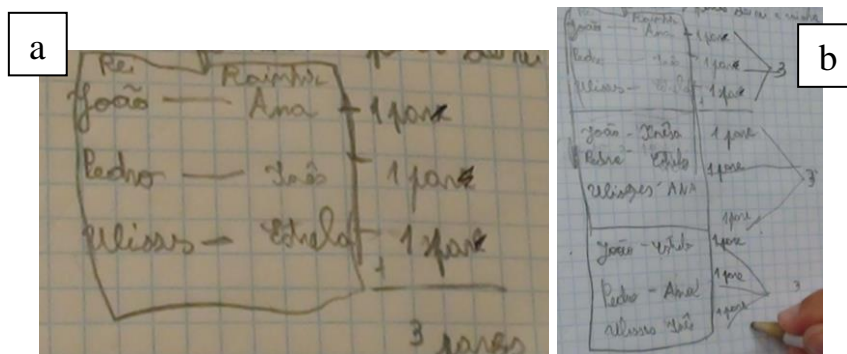
### **Introduction of the task**

Sofia reads the statement of the task, stressing the information that she finds important (number of boys and girls, awareness that a problem may have more than one answer), thus providing hints to the pupils. Noticing that some pupils struggle to understand the meaning of the verbal representation (the word "pair"), she challenges the pupils through open questioning ("Can I have two pairs and a half?", "What is a pair?"), but as the pupils remain silent, she decides to question them in a structured way ("How many persons do I have in a pair?"), getting an interpretation from one pupil ("A group of two!").

## Pupil's autonomous work

As Sofia notices that pupils are still struggling to find a strategy to solve the task, she decides to explore the task statement once more, addressing its key points. She gives hints and makes suggestions leading the pupils to review some of the task conditions like who can be Queen or King (“Who can be the King and the Queen?”, “Only one of the boys can be the King?”). She reinforces what she considers to be a complete solution (“So I want you tell me all the possibilities... All the ways of making a pair!”). To help the pupils to interpret the task statement she also suggests an active representation (“Imagine that . . . I am going to pick the King and Queen!... These three girls raise their arms . . . And these three boys want to be the King... And now... Which are the possibilities?”).

Sofia walks through the class, observing and questioning in detail almost all pupils. She challenges Angelo through open questioning to explain his mixed representation (“Can you explain me what is this? ...”). After noticing that the pupil has an incomplete answer (he says that there are three possible pairs – figure 1a) Sofia challenges him through open questioning (“Why João does not like Inês or Estrela? Is he angry with them?”), and then she informs him (“How many are the possibilities! It does not say: ‘Tell me three [possibilities]...”). When the pupil understands that his answer is incomplete, she lets him to continue to work autonomously. Later, Sofia comes back and challenges Angelo again, through open questioning, to explain his new mixed representation (figure 1b) (“What are you doing?”, “And what are you repeating here?”). Faced with his teacher’s challenge, Angelo explains to her why he now considers nine pairs and he easily describes his representation.



**Figure 1a e 1b – Angelo’s mixed representations before and after Sofia actions (verbal and iconic).**

Another pupil, Joaquim assumes that he has to use a pictorial representation (he is drawing every Queen and King) and starts to complain. Noticing that more pupils are also using pictorial representations, Sofia questions the class hinting them (“Did anyone told you: Spend a lot of time on drawings!? Or to draw all the Kings and Queens?”). As another pupil answers her questioning (“No! Why [should we draw]?! They have names!”) she reinforces that the pupils may choose freely their representa-

tion. Later, she returns to see how he is doing and she notices that Joaquim followed the advice of his colleague and he drew an mixed representation (figure 2).

João/Ana	João/Inês	João/Estrela
Pedro/Inês	Pedro/Ana	Pedro/Estrela
Ulisses/Estrela	Ulisses/Inês	Ulisses/Ana

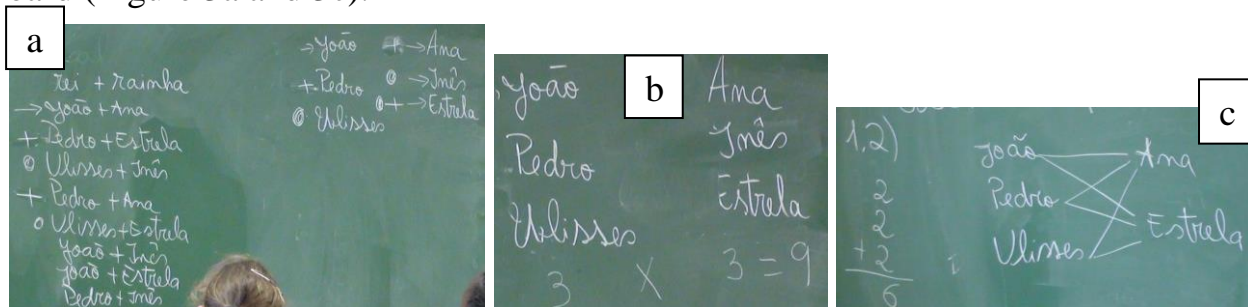
**Figure 2 – Joaquim’s mixed representation (verbal and iconic).**

This time, she challenges Joaquim (figure 2) through open questioning to explain his representation (“What are you doing?”) and he does it easily. During the pupil’s autonomous work, most solve the task by using different types of informal and preformal representations. At that time, Sofia decides to begin the whole class discussion.

### Whole class discussion

Sofia begins by inviting Luís to present his solution (he had an incomplete answer, as he indicated that there were six different pairs) and write it on the board. She asks him in a structured way to explain his representation (“Why did you not considered João and Estrela?”, “Can João be paired with someone else?”). During the discussion, through an iconic representation that Sofia made on the board, Luís and other pupils acknowledge that they forgot some pairs, and identify them easily (“Ah! He can [also be paired] with Ana!”).

Then, Sofia decides to pose to the whole class a follow up question (“If one of the girls drop out, how many pairs would be possible?”). This is a question that was solved during the autonomous work, only by the fastest pupils. One of those pupils, Laura, has no difficulty in presenting her answer and explaining to the class how she thought. Sofia then decides to transform Laura’s oral representation into a mixed and then a symbolic representation (figure 3c). At the same time, she tries to guide the pupils to establish connections between the representations that were written on the board (Figure 3a and 3c).



**Figure 3a,3b and 3c– Sofia’s iconic, symbolic and mixed representations.**

At the end of the discussion, Sofia introduces the multiplication sign (“If we have... Three boys [she writes “3” below the boys’ names] and three girls ([she writes “3”



below the girls' names]... I have (she puts the  $\times$  sign writing  $3 \times 3$ )... Nine! Nine possibilities!") (Figure 3b).

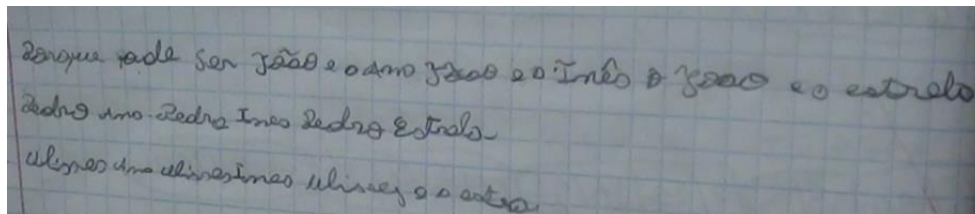
## SARA'S CLASS

## Introduction of the task

Sara challenges a pupil, André, to explain to her the statement of the task (“What did you get from the exercise?”). Faced with André’s difficulty in answering to her challenge, she hints him (“How many pairs... What is a pair?”). At a certain point she notices that the pupils are having difficulty in understanding the meaning of the verbal representation “pair” and she informs the pupils (“We need to have a King and a Queen!”). Afterwards she guides the pupils to focus into the information that she finds important (each pair must have a King and a Queen, who are the eligible boys and girls, there are several possible pairs). When the introduction of the task is almost finished, some pupils try to answer it orally without writing the answer (“I did it! It is...”!) and Sara reinforces the importance of writing and justifying all the answers in their notebook (“So do it!... In your exercise book!”, “I want you to explain me which are the pairs! And why!”).

## Pupil's autonomous work

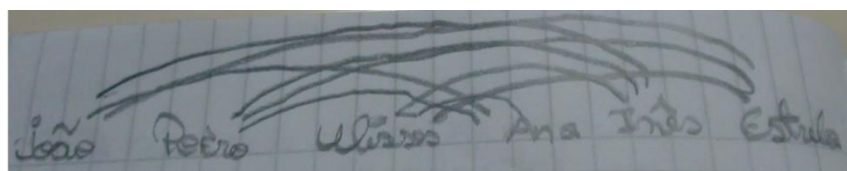
As some pupils try to answer Sara orally, she reinforces the importance of writing down their answer. Other pupils present their incomplete answers and she hints them, by saying “there are more pairs to be found”. Most pupils get the right answer by using a verbal representation similar to the answer of Carlos (Figure 4).



**Figure 1 - Verbal representation used by Carlos.**

Sara challenges Carlos to explain his representation (“And why? How did you saw it?”) and he does it easily. She continues to walk through the class and observes her pupils’ work. When she finds answers with different representations, she questions them with more detail.

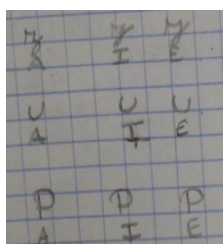
At some point Sara notices Mauro's mixed representation (Figure 5). She challenges him through open questioning to explain how he solved the task ("Explain it to me...") which he does with no difficulty. She praises his representation loudly ("Good work!") in order to induce other pupils to also find different representations.



**Figure 2 - Mixed representation (verbal and iconic) used by Mauro.**

After, Sara questions Mariana, the only pupil that uses a symbolic representation ( $3+3+3$  as a vertical computation) to solve the task. She challenges her to explain the representation (“I am not understanding [your representation]... Could you explain it to me?”). Most specifically, she wants to know if Mariana understands the meaning of each portion. As the pupil points to each portion and explains it (Ana with the three [boys] (points to the first line), Inês with three (points to the second line) and Estela with all three (points to the third line)... And it’s nine!!”), Sara is pleased with her answer and continues walking through the class.

Then Sara questions Leonardo, a pupil that felt compelled to find a “different representation” (Figure 6):



**Figure 3 – Leonardo’s mixed (verbal and iconic) representation**

Sara challenges Leonardo to explain his mixed representation which he does easily (“J” from João... “I” from Inês!... So... (as he points to each capital letter) Ana, Inês and Estrela. U is Ulisses... and Ana, Inês e Estrela! (points to P) This is Pedro with Ana, Inês e Estrela... Nine pairs!”). Then, Sara praises him loudly, and, once again, she tries to motivate other pupils to find different representations.

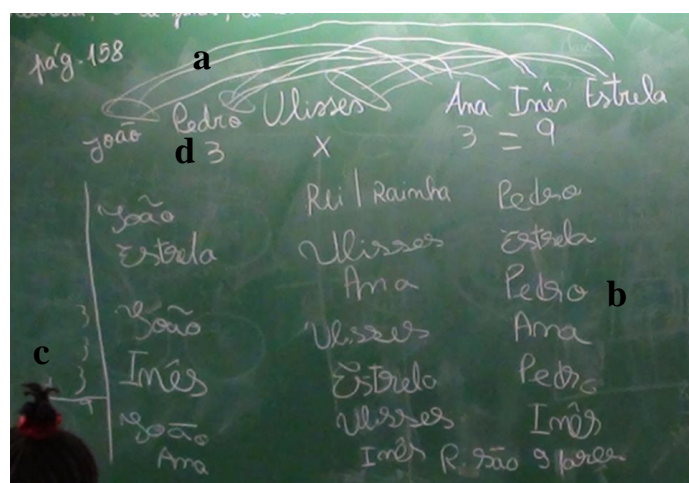
### **Whole class discussion**

Sara asks several pupils to present their answer to the class. The first is Jonas, a really shy and insecure pupil with whom Sara had been talking during pupils’ autonomous work, noticing that he had a right answer (figure 7). In the beginning of the whole class discussion Sara challenges Jonas to explain his answer (“Explain to me...”, “Why?”). However, faced with the difficulty of the pupil in answering, she decides to question him in a more structured way (“You did the pairs... Do you know why?”). She ends by guiding Jonas, giving him some information related to his first explanation (“You were trying to join a boy and a girl... Was it?”).

Afterwards, Sara challenges Mauro to show his answer (an iconic representation where he connects, in a scheme, the different characters’ names) (“How did you did that?”) and, sometimes she questions him in a more structured way (“What is that

[connection]?”). In the end of Mauro’s presentation, she guides the pupils in establishing connections between Jonas’ and Mauro’s representations.

The last pupil to present her answer is Mariana, who used a symbolic representation. This is also a very shy pupil and Sara begins by question her in a more structured way. Although Mariana explained perfectly her representation during pupils’ autonomous work, now she feels the need of using an active representation (counting her fingers) to assure that her answer is right. This leads Sara to change her actions and inform the class about Mariana’s explanation. Next, Sara teases pupils to catch their attention (“I am going to teach you a trick!”. When she starts talking it seems like she is guiding pupils to interpret the statement of the task (“How many boys?”, “How many girls?”). However, a glimpse of information (“Each boy can be in three pairs...”) is actually a challenge that triggers pupils to convert the presented representations into a symbolic representation of multiplication (“Teacher! There are three pairs of three!”, “It is three times three!”).



**Figure 7 – Mauro’s iconic representation (a), Jonas mixed representation (b), Mariana symbolic representation (c), and the class symbolic representation (d).**

Pleased with her pupils’ answers, Sara writes the symbolic representation ( $3 \times 3 = 9$ ) above Mauro’s representation (figure 7).

## CONCLUSION

In the introduction of the task, both teachers lead pupils in interpreting the statement of the task, focusing some key elements (number of boys and girls, characters names, main condition to have a pair). In both classes pupils struggle to interpret the meaning of the verbal representation “pair”, and both teachers felt the need of negotiating the meaning of “pair”. The main differences between Sofia and Sara concern their actions, as Sofia mainly hints through questioning (Who? How? How many?) and Sara often challenges her pupils.

During pupils’ autonomous work, Sofia and Sara (i) ask their pupils to write down their answers, despite the efforts of some to answer only orally; (ii) promote their



pupils' free choice of representations; and (iii) do not suggest alternatives nor guide their pupils to find conversions or treatments, even when they are struggling. Apparently, these actions would enable the emergence of a large variety of representations, but that does not happen in both classes. Thus, while Sofia's pupils use several types of representations (mainly informal and preformal), most pupils in Sara's class use an identical mixed representation and just a few use the symbolic representation of adding. The different results from their classes, seem to constrain the actions of Sofia and Sara. In Sofia's class, when a pupil shows her a wrong or incomplete answer she first challenges and questions the pupil, then she lets him to solve the task autonomously, and later she comes back to question that pupil again. In Sara's class, when a pupil shows her a wrong or incomplete answer she briefly advises him or her to review their answer. It seems that she is searching for pupils that are using different types of representations (as she also tries to motivate pupils to do that). When she finds someone that, according to her, has an interesting representation, she questions the pupil lingeringly, in order to understand if he or she is understanding his/her representation and is able to explain it.

In whole class discussions, both teachers register on the board all representations presented and that facilitates the establishment of connections between representations. Sofia and Sara also guide the pupils to establish connections between the representations presented and the symbolic representation of multiplication that no pupil has used during the autonomous work (Stylianou, 2010). As during pupils' autonomous work, teachers' actions in whole class discussions are also constrained by pupils' results and difficulties. That way, Sofia decides to ask a pupil with an incomplete answer to present his answer and then her actions are mainly informing, as she felt the need of guiding pupils to formal representations (her pupils used different types of representations but mainly informal and preformal ones) as in MacClain (2000). At the same time, Sara asked some key pupils to present their answers that included different representation types (her pupils used mainly the same iconic representation). At the end of whole class discussion, Sofia challenges pupils so they can find by themselves that  $3 \times 3$  is also a representation that can be used to answer the task.

During the class, the success of the task was influenced by the teachers' actions that changed according to pupils' activity (Swan, 2007). Regarding representations, Sofia and Sara moved towards more formal or more informal representations according to their perceptions of their pupils' difficulties. Regarding teachers' questioning, both tend to change their questions in what we may consider as a low or high level of challenge according to pupils' difficulties. That way, they usually started by challenging their pupils (a higher level of questioning) but, sometimes they felt that they had to decrease their questioning level into questioning in a more structured way.

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